Eur. Phys. J. C 27, 609–621 (2003)
Digital Object Identifier (DOI) 10.1140/epjc/s2002-01126-x

THE EUROPEAN
PHYSICAL JOURNAL C

On the coupling impedance for particles with opposite velocities

D. Davino[1], G. Dôme[2], L. Verolino[3]

[1] Università degli Studi del Sannio, Dipartimento di Ingegneria, Piazza Roma, 82100 Benevento, Italy
[2] Retired from CERN, 1211 Geneva 23, Switzerland
[3] Università Federico II di Napoli, Dipartimento di Ingegneria Elettrica, via Claudio 21, 80125 Napoli, Italy

Received: 27 March 2002 / Revised version: 5 December 2002 /
Published online: 26 February 2003 –

Abstract. As already shown by several authors, the reciprocity theorem can be rewritten as a simple formula connecting the longitudinal coupling impedances of a particle travelling in the positive and negative directions of an infinitely long vacuum chamber crossing a scattering structure (cavity, step junction, iris, etc.) of general shape. The formula is valid for any particle velocity. As an example, we consider the case of two semi-infinite circular vacuum chambers with different radii; for this case we give the explicit difference between the coupling impedances and the wake potentials.

1 Introduction

The aim of this paper is to find, by means of the reciprocity theorem, an explicit formula connecting the longitudinal coupling impedances of particles travelling in opposite directions.

We consider a general shape scatterer connected to two semi-infinite vacuum chambers of general and different shapes. Through the z-axis of such a structure a particle of charge q travels with a velocity[1] $v = \beta c$. The effects of the field radiated by the scatterer on the longitudinal motion of the particle can be described through the longitudinal coupling impedance. This impedance is a global parameter in the frequency domain, useful to describe the interaction between the particle and the surrounding structure. In the case of a particle of charge q it is defined by the integral [1]

$$-qZ_{\parallel}^{+}(k) = \int_{-\infty}^{+\infty} E_z^{\mathrm{s}+}(r=0,z)\mathrm{e}^{j(k/\beta)z}\mathrm{d}z,$$
$$qZ_{\parallel}^{-}(k) = \int_{-\infty}^{+\infty} E_z^{\mathrm{s}-}(r=0,z)\mathrm{e}^{-j(k/\beta)z}\mathrm{d}z, \qquad (1)$$

where the superscripts + and − correspond, respectively, to a particle travelling in the positive and negative z-axis direction, $E_z^{\mathrm{s}\pm}(r,z)$ is the radiated field, $k = \omega/c$ is the wavenumber, and β is taken as positive.

Several authors have worked on this subject. Heifets and Kheifets [2] gave a systematic review of theoretical results for the longitudinal and transverse impedances; they showed results for a step in circular beampipes in the case of ultrarelativistic particles ($\beta = 1$).

Kheifets and Zotter, in a book on wakes and coupling impedances [3], using published results [4–6], give a formula which is the same as our formula (17), but without deriving an explicit result. Recently, Heifets and Zotter [7] got another symmetry property of the longitudinal coupling impedance for off-axis particles in the case of structures with mirror symmetry with respect to z (which implies identical beampipes on both ends).

2 The reciprocity theorem

The Lorentz reciprocity theorem [8] is very useful in the solution of electromagnetic problems. As an example, it is often used to deduce some general properties of microwave devices and antennas. Let us introduce the expression of the reciprocity theorem. Consider an electric current source $\mathbf{J}^+$ located in a closed volume V and producing a field $\mathbf{E}^+$, $\mathbf{H}^+$ and another electric current source $\mathbf{J}^-$ acting in the same volume V and producing a field $\mathbf{E}^-$, $\mathbf{H}^-$. In such a case the reciprocity theorem reads, in frequency domain:

$$\oint_S (\mathbf{E}^+ \times \mathbf{H}^- - \mathbf{E}^- \times \mathbf{H}^+)\cdot \hat{n}\mathrm{d}S = \int_V (\mathbf{E}^- \cdot \mathbf{J}^+ - \mathbf{E}^+ \cdot \mathbf{J}^-)\mathrm{d}V \qquad (2)$$

where S is the bounding surface of the closed volume V and $\hat{n}$ is the unit outer normal to S.

3 The geometry and the fields

The geometry of our problem is as general as possible. As shown in Fig. 1, we consider a scatterer of general shape,

[1] In this paper c is the speed of light

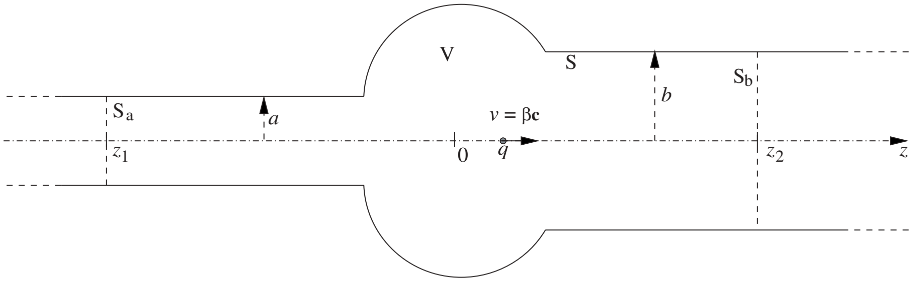

Fig. 1. A scatterer of general shape connecting two vacuum chambers

connected to a semi-infinite vacuum chamber of a certain transverse section on one side and to another semi-infinite vacuum chamber with a different transverse section on the other side. All the metallic surfaces are perfect conductors. Along the z-axis of such a structure a particle of charge q travels with a velocity $v = \beta c$.

If the particle runs in the positive direction of the z-axis ($0 < \beta \leq 1$), then the electric current density for the infinite waveguide in the frequency domain, using a cylindrical coordinate system, is

$$\mathbf{J}^{+}(r, z) = \hat{z}\operatorname{sgn}(v)\frac{q\delta(r)}{2\pi r}\mathrm{e}^{-j(k/\beta)z}, \tag{3}$$

where $\delta(r)$ is the Dirac delta function. If the particle runs in the opposite direction ($-1 \leq \beta < 0$) the current density is the opposite complex conjugate of the previous one, namely $\mathbf{J}^{-} = -(\mathbf{J}^{+})^{*}$.

Let us separate the fields into a primary field, which exists when the scatterer is not present, and a scattered field. If we assume that the dielectric is slightly lossy, the radiated field vanishes at $z = \pm\infty$.

In each waveguide, the total field can be described by the superposition of a field travelling with the charge itself and a radiated field which travels away from the scatterer, namely

$$\mathbf{E} = \mathbf{E}^{0} + \mathbf{E}^{s}, \quad \mathbf{H} = \mathbf{H}^{0} + \mathbf{H}^{s}.$$

The primary field $\mathbf{E}^0$, $\mathbf{H}^0$ is the field of a particle travelling in an infinite vacuum chamber similar to one or the other of the two semi-infinite chambers without discontinuity and scatterers, while the radiated field $\mathbf{E}^s$, $\mathbf{H}^s$ is due to the scatterer.

In the scatterer itself, $\mathbf{E}^0$ must be taken as the free-space field which travels with the particle. The coupling impedance involves only $\mathbf{E}^s$.

In the waveguides, $\mathbf{E}^0$ contains a contribution from the walls which corresponds to a coupling impedance per unit length; this contribution must be separated from $\mathbf{E}^s$ in order to avoid an infinite coupling impedance of the waveguides.

In the reciprocity theorem expression (2) we can consider $\mathbf{J}^{+}$ as the current due to a particle travelling in the positive direction and $\mathbf{J}^{-}$ as the opposite one. In such a case the fields are

$$\begin{aligned}\mathbf{E}^{+} &= \mathbf{E}^{0+} + \mathbf{E}^{\mathrm{s}+}, \quad \mathbf{H}^{+} = \mathbf{H}^{0+} + \mathbf{H}^{\mathrm{s}+},\\ \mathbf{E}^{-} &= \mathbf{E}^{0-} + \mathbf{E}^{\mathrm{s}-}, \quad \mathbf{H}^{-} = \mathbf{H}^{0-} + \mathbf{H}^{\mathrm{s}-}.\end{aligned}$$

In Sect. 5 we give, as an example, the primary field of a particle travelling in a circular waveguide.

In general, the scattered fields in the two vacuum chambers (we identify the left one with subscript a and the right one with subscript b) can be expanded into an orthogonal set of transverse modal vectors $\mathbf{E}_n(r,\varphi)$ and $\mathbf{H}_n(r,\varphi)$ which satisfy the metallic boundary conditions on the pipe walls, leading to

$$\begin{aligned}\mathbf{E}_a^{\mathrm{s}+}(r,\varphi,z) &= \sum_n a_n^{+}\mathbf{E}_n^{a}(r,\varphi)\mathrm{e}^{jk_n^a z},\\ \mathbf{H}_a^{\mathrm{s}+}(r,\varphi,z) &= \sum_n a_n^{+}\mathbf{H}_n^{a}(r,\varphi)\mathrm{e}^{jk_n^a z},\end{aligned} \tag{4}$$

$$\begin{aligned}\mathbf{E}_b^{\mathrm{s}+}(r,\varphi,z) &= \sum_n b_n^{+}\mathbf{E}_n^{b}(r,\varphi)\mathrm{e}^{-jk_n^b z},\\ \mathbf{H}_b^{\mathrm{s}+}(r,\varphi,z) &= \sum_n b_n^{+}\mathbf{H}_n^{b}(r,\varphi)\mathrm{e}^{-jk_n^b z},\end{aligned} \tag{5}$$

$$\begin{aligned}\mathbf{E}_a^{\mathrm{s}-}(r,\varphi,z) &= \sum_n a_n^{-}\mathbf{E}_n^{a}(r,\varphi)\mathrm{e}^{jk_n^a z},\\ \mathbf{H}_a^{\mathrm{s}-}(r,\varphi,z) &= \sum_n a_n^{-}\mathbf{H}_n^{a}(r,\varphi)\mathrm{e}^{jk_n^a z},\end{aligned} \tag{6}$$

$$\begin{aligned}\mathbf{E}_b^{\mathrm{s}-}(r,\varphi,z) &= \sum_n b_n^{-}\mathbf{E}_n^{b}(r,\varphi)\mathrm{e}^{-jk_n^b z},\\ \mathbf{H}_b^{\mathrm{s}-}(r,\varphi,z) &= \sum_n b_n^{-}\mathbf{H}_n^{b}(r,\varphi)\mathrm{e}^{-jk_n^b z}.\end{aligned} \tag{7}$$

We used the cylindrical coordinate system r, φ, but any transverse coordinate system can be used. These expressions are very general and involve both E and H modes. Furthermore, the general expression for the primary fields can be written as

$$\begin{aligned}\mathbf{E}^{0+}(r,\varphi,z) &= \boldsymbol{\mathcal{E}}^{0+}(r,\varphi)\mathrm{e}^{-j(k/\beta)z},\\ \mathbf{E}^{0-}(r,\varphi,z) &= \boldsymbol{\mathcal{E}}^{0-}(r,\varphi)\mathrm{e}^{j(k/\beta)z},\end{aligned} \tag{8}$$

$$\begin{aligned}\mathbf{H}^{0+}(r,\varphi,z) &= \boldsymbol{\mathcal{H}}^{0+}(r,\varphi)\mathrm{e}^{-j(k/\beta)z},\\ \mathbf{H}^{0-}(r,\varphi,z) &= \boldsymbol{\mathcal{H}}^{0-}(r,\varphi)\mathrm{e}^{j(k/\beta)z},\end{aligned} \tag{9}$$

where

$$\begin{aligned}&\mathcal{E}_z^{0-}(r,\varphi) = -\mathcal{E}_z^{0+}(r,\varphi), \qquad \mathcal{E}_r^{0-}(r,\varphi) = \mathcal{E}_r^{0+}(r,\varphi),\\ &\mathcal{H}_\varphi^{0-}(r,\varphi) = -\mathcal{H}_\varphi^{0+}(r,\varphi).\end{aligned} \tag{10}$$

4 Solution

Let us start with the right-hand side of the reciprocity theorem in expression (2), in which we substitute the total field with the superposition of primary and radiated field:

$$\int_V [(\mathbf{E}^{0-} + \mathbf{E}^{\mathrm{s}-}) \cdot \mathbf{J}^+ - (\mathbf{E}^{0+} + \mathbf{E}^{\mathrm{s}+}) \cdot \mathbf{J}^-] \mathrm{d}V. \quad (11)$$

As shown in Fig. 1, we choose as volume V the volume enclosed by the inner surface of the general scatterer plus the inner surfaces of the two vacuum chambers, closed by the two cross-sections of vacuum chambers at abscissae $z_1 < 0$ and $z_2 > 0$, respectively. Making use of (3), the right-hand side of expression (2) becomes

$$q \int_{z_1}^{z_2} \Big\{ [E_z^{0-}(r=0,z) + E_z^{\mathrm{s}-}(r=0,z)] \mathrm{e}^{-j(k/\beta)z} + [E_z^{0+}(r=0,z) + E_z^{\mathrm{s}+}(r=0,z)] \mathrm{e}^{j(k/\beta)z} \Big\} \mathrm{d}z.$$

Since, following (10), E_z^0 changes sign with v, the contribution of the primary fields[2] is zero in the scatterer as well as in the waveguides. Therefore the previous expression reduces to

$$q \int_{z_1}^{z_2} [E_z^{\mathrm{s}-}(r=0,z) \mathrm{e}^{-j(k/\beta)z} + E_z^{\mathrm{s}+}(r=0,z) \mathrm{e}^{j(k/\beta)z}] \mathrm{d}z.$$

With the definitions (1), this expression can be rewritten as

$$\begin{aligned} & q^2 [Z_\parallel^-(k) - Z_\parallel^+(k)] \\ & -q \int_{-\infty}^{z_1} [E_z^{\mathrm{s}-}(r=0,z) \mathrm{e}^{-j(k/\beta)z} \\ & \quad + E_z^{\mathrm{s}+}(r=0,z) \mathrm{e}^{j(k/\beta)z}] \mathrm{d}z \\ & -q \int_{z_2}^{\infty} [E_z^{\mathrm{s}-}(r=0,z) \mathrm{e}^{-j(k/\beta)z} \\ & \quad + E_z^{\mathrm{s}+}(r=0,z) \mathrm{e}^{j(k/\beta)z}] \mathrm{d}z. \end{aligned} \quad (12)$$

This should be equal to the left-hand side of (2) where

$$\begin{aligned} & (\mathbf{E}^+ \times \mathbf{H}^- - \mathbf{E}^- \times \mathbf{H}^+) \cdot \hat{n} \\ & = \overbrace{(\mathbf{E}^{0+} \times \mathbf{H}^{0-} - \mathbf{E}^{0-} \times \mathbf{H}^{0+}) \cdot \hat{n}}^{\mathbf{S}^0} \\ & + \overbrace{(\mathbf{E}^{\mathrm{s}+} \times \mathbf{H}^{\mathrm{s}-} - \mathbf{E}^{\mathrm{s}-} \times \mathbf{H}^{\mathrm{s}+}) \cdot \hat{n}}^{\mathbf{S}^{\mathrm{s}}} \quad (13) \\ & + \underbrace{(\mathbf{E}^{0+} \times \mathbf{H}^{\mathrm{s}-} - \mathbf{E}^{\mathrm{s}-} \times \mathbf{H}^{0+} + \mathbf{E}^{\mathrm{s}+} \times \mathbf{H}^{0-} - \mathbf{E}^{0-} \times \mathbf{H}^{\mathrm{s}+}) \cdot \hat{n}}_{\mathbf{S}^{0\mathrm{s}}}. \end{aligned}$$

These three terms have to be integrated over the whole bounding surface S. Let us explicitly note that the term $\mathbf{S}^0$ is produced only by the primary field, whereas in $\mathbf{S}^{\mathrm{s}}$ and $\mathbf{S}^{0\mathrm{s}}$ the scattered field is also involved.

Note that all the surface integrals on the (perfectly conducting) bounding surfaces of the vacuum chambers and of the scatterer are zero, since they involve the tangential component of electric fields. Only the integrals on the two cross-sections of the waveguides at abscissae z_1 and z_2 give a contribution; let us call these surfaces S_a and S_b. Using expressions (4) to (7) and expressions (8) and (9):

$$\begin{aligned} \oint \mathbf{S}^{\mathrm{s}} \mathrm{d}S = & \Big[-\sum_n a_n^+ \sum_m a_m^- + \sum_n a_n^- \sum_m a_m^+ \Big] \\ & \times \int_{S_a} [\mathbf{E}_n^a(r,\varphi) \times \mathbf{H}_m^a(r,\varphi)] \cdot \hat{z} \mathrm{d}S \mathrm{e}^{j(k_n^a + k_m^a) z_1} \\ & + \Big[\sum_n b_n^+ \sum_m b_m^- - \sum_n b_n^- \sum_m b_m^+ \Big] \\ & \times \int_{S_b} [\mathbf{E}_\mathrm{n}^b(r,\varphi) \times \mathbf{H}_m^b(r,\varphi)] \cdot \hat{z} \mathrm{d}S \mathrm{e}^{-j(k_n^b + k_m^b) z_2}, \end{aligned} \quad (14)$$

$$\begin{aligned} \oint \mathbf{S}^{0\mathrm{s}} \mathrm{d}S = & \Big[-\sum_n a_n^- \int_{S_a} [\boldsymbol{\mathcal{E}}_a^{0+}(r,\varphi) \times \mathbf{H}_n^a(r,\varphi)] \cdot \hat{z} \mathrm{d}S \\ & + \sum_n a_n^- \int_{S_a} [\mathbf{E}_n^a(r,\varphi) \times \boldsymbol{\mathcal{H}}_a^{0+}(r,\varphi)] \cdot \hat{z} \mathrm{d}S \Big] \mathrm{e}^{-j((k/\beta) - k_n^a) z_1} \\ & + \Big[-\sum_n a_n^+ \int_{S_a} [\mathbf{E}_n^a(r,\varphi) \times \boldsymbol{\mathcal{H}}_a^{0-}(r,\varphi)] \cdot \hat{z} \mathrm{d}S \\ & + \sum_n a_n^+ \int_{S_a} [\mathcal{E}_a^{0-}(r,\varphi) \times \mathbf{H}_n^a(r,\varphi)] \cdot \hat{z} \mathrm{d}S \Big] \mathrm{e}^{j((k/\beta) + k_n^a) z_1} \\ & + \Big[\sum_n b_n^- \int_{S_b} [\mathcal{E}_b^{0+}(r,\varphi) \times \mathbf{H}_n^b(r,\varphi)] \cdot \hat{z} \mathrm{d}S \\ & - \sum_n b_n^- \int_{S_b} [\mathbf{E}_n^b(r,\varphi) \times \boldsymbol{\mathcal{H}}_b^{0+}(r,\varphi)] \cdot \hat{z} \mathrm{d}S \Big] \mathrm{e}^{-j((k/\beta) + k_n^b) z_2} \\ & + \Big[\sum_n b_n^+ \int_{S_b} [\mathbf{E}_n^b(r,\varphi) \times \boldsymbol{\mathcal{H}}_b^{0-}(r,\varphi)] \cdot \hat{z} \mathrm{d}S \quad (15) \\ & - \sum_n b_n^+ \int_{S_b} [\mathcal{E}_b^{0-}(r,\varphi) \times \mathbf{H}_n^b(r,\varphi)] \cdot \hat{z} \mathrm{d}S \Big] \mathrm{e}^{j((k/\beta) - k_n^b) z_2}. \end{aligned}$$

It is apparent that both the $\mathbf{S}^{\mathrm{s}}$ and $\mathbf{S}^{0\mathrm{s}}$ terms have an exponential factor which explicitly depends on the abscissae z_1 and z_2; in the limit $|z_1|, z_2 \to \infty$ these terms vanish. This is obvious for the evanescent modes, when k_n is imaginary. When k_n is real, we can justify this assertion with two types of considerations.

(1) Consider that the dielectric filling the volume V has very small losses, the amount of which is as little as we want. This assumption does not change all the computation but establishes that any contribution which depends on the abscissae z_1 or z_2 must vanish when z_1 or z_2 goes to $\pm\infty$, through a small imaginary part of k_n^a or k_n^b.

(2) The contributions to the coupling impedance of the terms $\mathbf{S}^{\mathrm{s}}$ and $\mathbf{S}^{0\mathrm{s}}$ depend on $\mathrm{e}^{jk_\alpha z_{1,2}}$ where k_α represents a generic wavenumber. It is apparent that a local average over $z_{1,2}$ of these contributions is zero, that is $\langle \mathrm{e}^{jk_\alpha z_{1,2}} \rangle = 0$ when k_α is real.

Nevertheless, in Appendix A we give an explicit proof that the exponentials in (12) and (15) are identical, and

[2] Note that the primary fields E_z^0 are divergent as $\ln r$ for $r \to 0$; see an example of primary fields in (20) and (23)

therefore cancel each other exactly on both sides of (2). Regarding (14), when integrated over S, the $\mathbf{S}^{\mathrm{s}}$ term vanishes completely because of the orthogonality relations [9]:

$$\int_S [\mathbf{E}_n \times \mathbf{H}_m]_z \mathrm{d}S = \delta_{nm} \int_S [\mathbf{E}_n \times \mathbf{H}_n]_z \mathrm{d}S. \quad (16)$$

Therefore, in general we have

$$\begin{aligned} &q^2[Z_\parallel^-(k) - Z_\parallel^+(k)] \\ &= \int_{S_a} (\mathbf{E}^{0+} \times \mathbf{H}^{0-} - \mathbf{E}^{0-} \times \mathbf{H}^{0+}) \cdot \hat{n} \mathrm{d}S \\ &+ \int_{S_b} (\mathbf{E}^{0+} \times \mathbf{H}^{0-} - \mathbf{E}^{0-} \times \mathbf{H}^{0+}) \cdot \hat{n} \mathrm{d}S, \quad (17) \end{aligned}$$

and the difference between coupling impedances involves only integrals of the primary fields which are independent of z_1 and z_2. It is easy to see [3] that in a perfectly conducting waveguide

$$\mathbf{E}^{0-} = \left(\mathbf{E}^{0+}\right)^*, \quad \mathbf{H}^{0-} = -\left(\mathbf{H}^{0+}\right)^*. \quad (18)$$

Therefore (17) can also be rewritten as

$$\begin{aligned} &q^2[Z_\parallel^-(k) - Z_\parallel^+(k)] \\ &= -\int_{S_a} 2\,\mathrm{Re}[\mathbf{E}^{0+} \times (\mathbf{H}^{0+})^*] \cdot \hat{n} \mathrm{d}S \\ &- \int_{S_b} 2\,\mathrm{Re}[\mathbf{E}^{0+} \times (\mathbf{H}^{0+})^*] \cdot \hat{n} \mathrm{d}S. \quad (19) \end{aligned}$$

5 Circular vacuum chambers

For the sake of clarity, and to reach a final closed formula, let us now restrict ourselves to the case of circular waveguides. Let us consider a particle of charge q travelling with a constant velocity $v = \beta c$ along the axis of a perfectly conducting circular waveguide of radius a. The electromagnetic field sustained by this particle exhibits a E_0 mode structure with azimuthal symmetry, having only radial and longitudinal components for the electric field, and an azimuthal component for the magnetic field. Choosing a cylindrical reference frame coaxial to the waveguide, we have [10]

$$\begin{aligned} E_{az}^{0+}(r,z) &= \mathrm{sgn}(\omega)\frac{jq\zeta_0\kappa}{2\pi\beta\gamma} \\ &\times \left[K_0(\kappa r) - I_0(\kappa r)\frac{K_0(\kappa a)}{I_0(\kappa a)}\right] \mathrm{e}^{-j(k/\beta)z}, \quad (20) \\ E_{ar}^{0+}(r,z) &= \frac{q\zeta_0\kappa}{2\pi|\beta|} \\ &\times \left[K_1(\kappa r) + I_1(\kappa r)\frac{K_0(\kappa a)}{I_0(\kappa a)}\right] \mathrm{e}^{-j(k/\beta)z}, \quad (21) \\ H_{a\varphi}^{0+}(r,z) &= \mathrm{sgn}(\beta)\frac{q\kappa}{2\pi} \\ &\times \left[K_1(\kappa r) + I_1(\kappa r)\frac{K_0(\kappa a)}{I_0(\kappa a)}\right] \mathrm{e}^{-j(k/\beta)z}, \quad (22) \end{aligned}$$

where $\gamma = (1-\beta^2)^{-1/2}$ is the Lorentz factor, $\kappa = |k/(\beta\gamma)|$ is the normalized wavenumber, $\zeta_0 = (\mu_0/\epsilon_0)^{1/2}$ is the impedance of free space, and $I_n(x)$ and $K_n(x)$ are modified Bessel functions of the first and second kind of order n, respectively. It is worth noting that

$$\frac{E_{ar}^{0+}(r,z)}{H_{a\varphi}^{0+}(r,z)} = \frac{\zeta_0}{\beta}.$$

For a particle travelling in the negative direction of the z-axis,

$$\begin{aligned} E_{az}^{0-}(r,z) &= -\mathrm{sgn}(\omega)\frac{jq\zeta_0\kappa}{2\pi\beta\gamma} \\ &\times \left[K_0(\kappa r) - I_0(\kappa r)\frac{K_0(\kappa a)}{I_0(\kappa a)}\right] \mathrm{e}^{j(k/\beta)z}, \quad (23) \\ E_{ar}^{0-}(r,z) &= \frac{q\zeta_0\kappa}{2\pi\beta} \\ &\times \left[K_1(\kappa r) + I_1(\kappa r)\frac{K_0(\kappa a)}{I_0(\kappa a)}\right] \mathrm{e}^{j(k/\beta)z}, \quad (24) \\ H_{a\varphi}^{0-}(r,z) &= -\frac{q\kappa}{2\pi} \\ &\times \left[K_1(\kappa r) + I_1(\kappa r)\frac{K_0(\kappa a)}{I_0(\kappa a)}\right] \mathrm{e}^{j(k/\beta)z}. \quad (25) \end{aligned}$$

In the formulae (23)–(25) β is taken as positive.

In the limit $a \to \infty$, we have the well-known field of a charged particle travelling in free space:

$$E_z^{0+}(r,z) = \mathrm{sgn}(\omega)\frac{jq\zeta_0\kappa}{2\pi\beta\gamma} K_0(\kappa r)\mathrm{e}^{-j(k/\beta)z}, \quad (26)$$

$$E_r^{0+}(r,z) = \frac{q\zeta_0\kappa}{2\pi|\beta|} K_1(\kappa r)\mathrm{e}^{-j(k/\beta)z}, \quad (27)$$

$$H_\varphi^{0+}(r,z) = \mathrm{sgn}(\beta)\frac{q\kappa}{2\pi} K_1(\kappa r)\mathrm{e}^{-j(k/\beta)z}. \quad (28)$$

If the particle moves at the speed of light, namely $|\beta| = 1$, all the previous formulae simplify. Because of the behavior of the modified Bessel functions for small arguments [11], the fields of a particle moving on the axis of an infinite waveguide exhibit a TEM mode structure:

$$\begin{aligned} &E_z^{0+}(r,z) = 0, \quad E_r^{0+}(r,z) = \frac{q\zeta_0}{2\pi r}\mathrm{e}^{-jkz\mathrm{sgn}(\beta)}, \\ &H_\varphi^{0+}(r,z) = \mathrm{sgn}(\beta)\frac{q}{2\pi r}\mathrm{e}^{-jkz\mathrm{sgn}(\beta)}. \quad (29) \end{aligned}$$

In the case of a circular vacuum chamber, the integration of the $\mathbf{S}^0$ term in (17) leads to

$$\int_{S_a} \mathbf{S}^0 \mathrm{d}S = \int_{S_a} [-\mathcal{E}_r^{0a+}(r)\mathcal{H}_\varphi^{0a-}(r) + \mathcal{E}_r^{0a-}(r)\mathcal{H}_\varphi^{0a+}(r)]\mathrm{d}S,$$

$$\int_{S_b} \mathbf{S}^0 \mathrm{d}S = \int_{S_b} [\mathcal{E}_r^{0b+}(r)\mathcal{H}_\varphi^{0b-}(r) - \mathcal{E}_r^{0b-}(r)\mathcal{H}_\varphi^{0b+}(r)]\mathrm{d}S,$$

where, for example, according to (8), the term $\mathcal{E}_r^{0a+}(r)$ designates the primary field expression without the exponential factor. Substituting the primary field expressions

(20)–(25) we obtain

$$\int_{S_a} \mathbf{S}^0 \mathrm{d}S + \int_{S_b} \mathbf{S}^0 \mathrm{d}S = 2\pi \frac{\zeta_0}{|\beta|} \left(\frac{q}{2\pi}\right)^2 2\kappa^2 \times \left\{ \int_0^a \left[K_1(\kappa r) + I_1(\kappa r)\frac{K_0(\kappa a)}{I_0(\kappa a)}\right]^2 r\mathrm{d}r - \int_0^b \left[K_1(\kappa r) + I_1(\kappa r)\frac{K_0(\kappa b)}{I_0(\kappa b)}\right]^2 r\mathrm{d}r \right\}. \quad (30)$$

It is apparent that we need the following type of integrals [12]:

$$\begin{aligned} &2\kappa^2 \int_a^\infty [K_1(\kappa r)]^2 r\mathrm{d}r \\ &= -[\kappa a K_1(\kappa a)]^2 + [\kappa a K_0(\kappa a)]^2 + 2\kappa a K_1(\kappa a) K_0(\kappa a), \\ &2\kappa^2 \int_0^a [I_1(\kappa r)]^2 r\mathrm{d}r \\ &= [\kappa a I_1(\kappa a)]^2 - [\kappa a I_0(\kappa a)]^2 + 2\kappa a I_1(\kappa a) I_0(\kappa a), \\ &2\kappa^2 \int_0^a K_1(\kappa r) I_1(\kappa r) r\mathrm{d}r \\ &= (\kappa a)^2 K_1(\kappa a) I_1(\kappa a) + (\kappa a)^2 K_0(\kappa a) I_0(\kappa a) \\ &- 2\kappa a I_1(\kappa a) K_0(\kappa a). \end{aligned} \quad (31)$$

In (30),

$$\int_0^a [K_1(\kappa r)]^2 r\mathrm{d}r = \int_0^\infty [K_1(\kappa r)]^2 r\mathrm{d}r - \int_a^\infty [K_1(\kappa r)]^2 r\mathrm{d}r, \quad (32)$$

where $\int_0^\infty [K_1(\kappa r)]^2 r\mathrm{d}r$ is infinite but independent of a; this integral represents the (infinite) power flux of a point charge in free space.

Finally, making use of (17), after algebraic manipulations we obtain

$$Z_\parallel^-(k) - Z_\parallel^+(k) = \frac{\zeta_0}{2\pi|\beta|} \left\{ \frac{2K_0(\kappa\ell)}{I_0(\kappa\ell)} - \frac{1}{[I_0(\kappa\ell)]^2} \right\}_{\ell=a}^{\ell=b}, \quad (33)$$

which is a general expression valid for any frequency and particle velocity. The function which appears in the curly brackets of expression (33) is plotted in Fig. 2.

Note that the right-hand side of (33) is real and this means that the imaginary parts of impedances are always equal, namely

$$\mathrm{Im}[Z_\parallel^-(k)] = \mathrm{Im}[Z_\parallel^+(k)].$$

Furthermore, it is apparent that if the vacuum chamber radii are equal ($a = b$), then the real parts of impedances for particles travelling in opposite directions are also equal, namely $\mathrm{Re}[Z_\parallel^-(k)] = \mathrm{Re}[Z_\parallel^+(k)]$, in spite of any existing scatterer.

Figure 3 shows the longitudinal coupling impedance of a step junction between two waveguides. In this figure

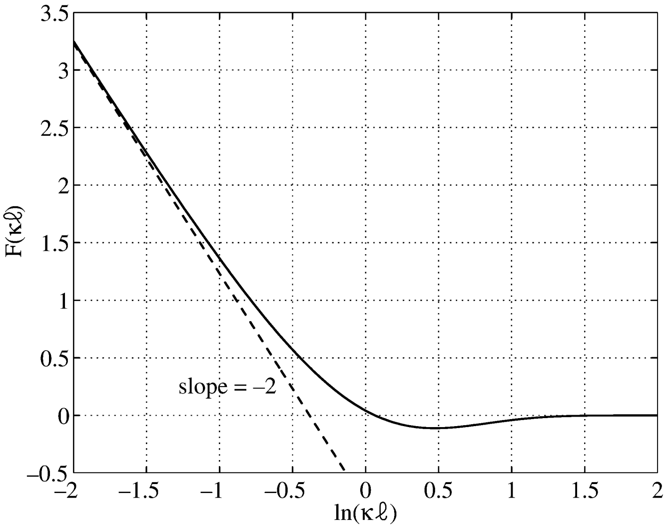

Fig. 2. The function $F(\kappa\ell) = \left[2K_0(\kappa\ell)/I_0(\kappa\ell) - 1/[I_0(\kappa\ell)]^2\right]$ versus ln $(\kappa\ell)$, where $\ell = a$ or b. For $\kappa\ell \gg 1$, the asymptotic behavior is $2\pi(1-\kappa\ell)\mathrm{e}^{-2\kappa\ell}$

Z^+ is the so called step-out case, whereas Z^- is the step-in case. We get the numerical results for both cases by using a standard mode matching technique [2,10,13]; afterwards we compute the difference and compare it to the theoretical formula (33).

The numerical computations have been made with MatLab, N_a modes being used in waveguide a, and N_b modes in waveguide b. The relative error is small as long as the absolute values of the impedances are not too small, even though the total number of modes used, $N_a + N_b = 362$, is not particularly large.

5.1 Limiting cases

The previous expression (33) simplifies very much when the particle runs at light velocity, namely $|\beta| = 1$, or when, for any velocity, we approach the limit $k \to 0$.

In the first case the expressions (20)–(25) for the primary fields become

$$\begin{aligned} &E_z^{0+}(r,z) = 0, \quad E_r^{0+}(r,z) = \frac{q\zeta_0}{2\pi r}\mathrm{e}^{-jkz}, \\ &H_\varphi^{0+}(r,z) = \frac{q}{2\pi r}\mathrm{e}^{-jkz}, \\ &E_z^{0-}(r,z) = 0, \quad E_r^{0-}(r,z) = \frac{q\zeta_0}{2\pi r}\mathrm{e}^{jkz}, \\ &H_\varphi^{0-}(r,z) = -\frac{q}{2\pi r}\mathrm{e}^{jkz}. \end{aligned} \quad (34)$$

We explicitly note that there is no longer any dependence on the vacuum chamber radius. So, we obtain

$$Z_\parallel^- - Z_\parallel^+ = -\frac{\zeta_0}{\pi}\ln(b/a) \quad (35)$$

which is constant and independent of wavenumber k. This result is reported in several papers of the literature; see for example [2].

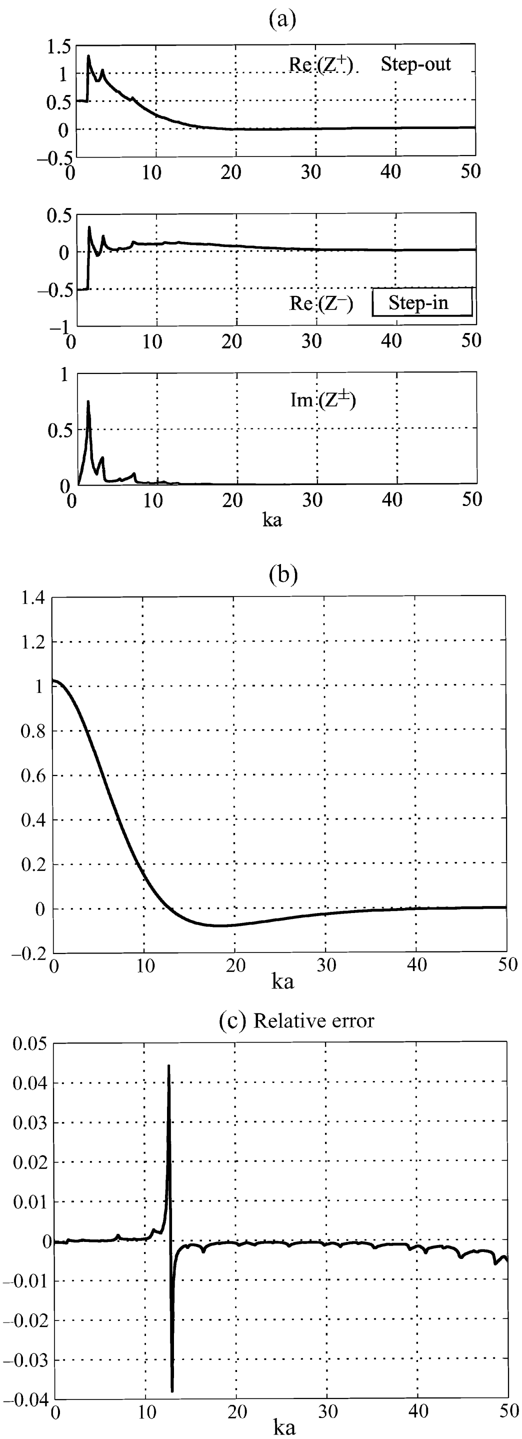

Fig. 3. (**a**) The normalized coupling impedances [normalized to $\zeta_0/(2\pi)$] for a step between two circular waveguides, evaluated by a mode matching code. The total number of modes is $N_a + N_b = 362$. The real part of the impedance is shown for the step-out and the step-in cases; the imaginary part is the same for both cases. (**b**) The difference of the coupling impedances $Z^+ - Z^-$ evaluated by the analytical formula (33) [normalized to $\zeta_0/(2\pi)$]. The value at the origin is $2\ln(b/a)$. (**c**) The relative error between (**a**) and (**b**), computed as $2((\mathbf{a}) - (\mathbf{b}))/((\mathbf{a}) + (\mathbf{b}))$. In all cases $\beta\gamma = 10$ and $a/b = 0.6$. Note that Z^+ is the step-out case, when the particle travels from the small to the large waveguide

If the wavenumber $k \to 0$, the primary fields are slightly different from the previous ones:

$$
\begin{aligned}
&E_z^{0+}(r,z) = 0, \quad E_r^{0+}(r,z) = \frac{q\zeta_0}{2\pi|\beta|r}, \\
&H_\varphi^{0+}(r,z) = \frac{q}{2\pi r}, \\
&E_z^{0-}(r,z) = 0, \quad E_r^{0-}(r,z) = \frac{q\zeta_0}{2\pi|\beta|r}, \\
&H_\varphi^{0-}(r,z) = -\frac{q}{2\pi r},
\end{aligned} \tag{36}
$$

and in this case

$$
Z_\parallel^- - Z_\parallel^+ = -\frac{\zeta_0}{\pi|\beta|}\ln(b/a), \tag{37}
$$

which is the difference at $k = 0$. Of course, we recover the previous limiting case when $|\beta| = 1$.

6 Wake potentials

In the present section we deduce the analytical expression for the difference of wake potentials by taking the inverse Fourier transform of formula (33), which reads

$$
\begin{aligned}
W^-(\tau) - W^+(\tau) = &\frac{1}{2\pi\epsilon_0|\beta|} \\
&\times \int_0^\infty \frac{1}{\pi}\left\{\frac{2K_0(k\ell)}{I_0(k\ell)} - \frac{1}{[I_0(k\ell)]^2}\right\}_{\ell=a/|\beta\gamma|}^{\ell=b/|\beta\gamma|} \cos(kc\tau)\mathrm{d}k.
\end{aligned} \tag{38}
$$

In this equation we mainly have to Fourier transform the function of k

$$
F(k\ell) = \frac{2K_0(k\ell)}{I_0(k\ell)} - \frac{1}{[I_0(k\ell)]^2}, \tag{39}
$$

where now $\ell = a/|\beta\gamma|$ or $\ell = b/|\beta\gamma|$. The inverse Fourier transform of (39) is $f(c\tau, \ell)$ with

$$
\begin{aligned}
\pi f(c\tau,\ell) &= \int_0^\infty F(k\ell)\cos(kc\tau)\mathrm{d}k \\
&= \int_0^\infty 2\cos(kc\tau)\frac{K_0(k\ell)}{I_0(k\ell)}\mathrm{d}k - \int_0^\infty \frac{\cos(kc\tau)}{[I_0(k\ell)]^2}\mathrm{d}k.
\end{aligned} \tag{40}
$$

Noting that

$$
\int_0^\infty \cos(kc\tau)\frac{K_0(k\ell)}{I_0(k\ell)}\mathrm{d}k = \frac{1}{c\tau}\int_0^\infty \frac{\sin(kc\tau)}{k[I_0(k\ell)]^2}\mathrm{d}k
$$

and introducing the auxiliary function

$$
u(c\tau,\ell) = \int_0^\infty \frac{\sin(kc\tau)}{k[I_0(k\ell)]^2}\mathrm{d}k, \tag{41}
$$

(40) becomes

$$
\pi f(c\tau,\ell) = \frac{2}{c\tau}u(c\tau,\ell) - u'(c\tau,\ell), \tag{42}
$$

where

$$u'(c\tau,\ell) = \frac{\partial u}{\partial(c\tau)} \qquad \text{and} \qquad u(0,\ell)=0. \tag{43}$$

From (42) and (41) we note that

$$\pi f(0,\ell) = \lim_{\tau\to 0}\frac{u(c\tau,\ell)}{c\tau} = \frac{1}{\ell}\int_0^\infty \frac{\mathrm{d}x}{[I_0(x)]^2} = \frac{\pi}{\ell}\frac{A}{2}. \tag{44}$$

The numerical value of the integral can be obtained by Mathematica as

$$A = \frac{2}{\pi}\int_0^\infty \frac{\mathrm{d}x}{[I_0(x)]^2} = 0.870690132537944. \tag{45}$$

The function $1/[I_0(x)]^2$, as will be shown in Appendix C, can be expanded in a series [14] as

$$\frac{1}{[I_0(x)]^2} = 1 - 4x^2\sum_{n=1}^{\infty}\frac{1}{[J_1(j_n)]^2(x^2+j_n^2)^2}, \tag{46}$$

where the j_n are the zeros of the Bessel function of order zero, namely $J_0(j_n)=0\ \forall n$.

After an integration term by term, the auxiliary function (41) becomes

$$u(c\tau,\ell) = \mathrm{sgn}(c\tau)\left[\frac{\pi}{2} - \frac{\pi c|\tau|}{\ell}\sum_{n=1}^{\infty}\frac{\mathrm{e}^{-c|\tau|j_n/\ell}}{j_n[J_1(j_n)]^2}\right], \tag{47}$$

and, as a consequence, $u'(c\tau,\ell)$ reads

$$u'(c\tau,\ell) = -\frac{\pi}{\ell}\left[\sum_{n=1}^{\infty}\frac{\mathrm{e}^{-c|\tau|j_n/\ell}}{j_n[J_1(j_n)]^2} - \frac{c|\tau|}{\ell}\sum_{n=1}^{\infty}\frac{\mathrm{e}^{-c|\tau|j_n/\ell}}{[J_1(j_n)]^2}\right]. \tag{48}$$

There is no Dirac function $\delta(c\tau)$ in $u'(c\tau,\ell)$ because $u(0,\ell)=0$. Therefore, from (42) we can write

$$f(c\tau,\ell) = \frac{1}{c|\tau|} - \frac{1}{\ell}\left[\sum_{n=1}^{\infty}\frac{\mathrm{e}^{-c|\tau|j_n/\ell}}{j_n[J_1(j_n)]^2} + \frac{c|\tau|}{\ell}\sum_{n=1}^{\infty}\frac{\mathrm{e}^{-c|\tau|j_n/\ell}}{[J_1(j_n)]^2}\right]. \tag{49}$$

It is apparent that the expression $\ell f(c\tau,\ell)$ is a universal function of $c|\tau|/\ell$, which is plotted in Fig. 4.

Using (40) with (49) in (38) immediately yields the analytic expression for the difference of the wake potentials as

$$W^-(\tau) - W^+(\tau) = \frac{1}{2\pi\epsilon_0|\beta|}\left\{\frac{1}{c|\tau|} - \frac{1}{\ell}\left[\sum_{n=1}^{\infty}\frac{\mathrm{e}^{-c|\tau|j_n/\ell}}{j_n[J_1(j_n)]^2} + \frac{c|\tau|}{\ell}\sum_{n=1}^{\infty}\frac{\mathrm{e}^{-c|\tau|j_n/\ell}}{[J_1(j_n)]^2}\right]\right\}_{\ell=a/|\beta\gamma|}^{\ell=b/|\beta\gamma|}. \tag{50}$$

Note that the expansion (49) has a very fast convergence if the argument $c|\tau|/\ell$ is not too small, while a small $c|\tau|/\ell$ leads to a very slow convergence. In the latter case

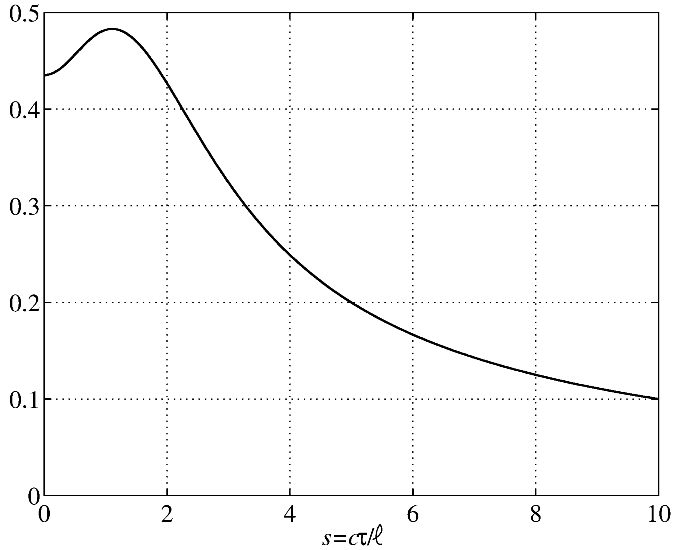

Fig. 4. The universal function $\ell f(c\tau,\ell)$ versus $s = c\tau/\ell$; here $\ell = a/|\beta\gamma|$ or $b/|\beta\gamma|$. The value at the origin is $A/2 = 0.435345$. For $s > 5$, this function is equal to ($1/s$ − fast decreasing exponentials)

it is possible to speed up the convergence by using the asymptotic expansions of the Bessel function $J_1(j_n)$ and of the j_n [11]:

$$J_1(j_n) \approx \sqrt{\frac{2}{\pi j_n}}, \quad j_n \approx (n-1/4)\pi,$$

and therefore, letting

$$\xi = \frac{\pi}{2}\left|\frac{c\tau}{\ell}\right| : \tag{51}$$

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{\mathrm{e}^{-c|\tau|j_n/\ell}}{j_n[J_1(j_n)]^2} &= \sum_{n=1}^{\infty}\frac{\mathrm{e}^{-j_n(2/\pi)\xi}}{j_n[J_1(j_n)]^2} \approx \frac{\pi}{2}\sum_{n=1}^{\infty}\mathrm{e}^{-(n-1/4)2\xi}\\
&= \frac{\pi}{2}\frac{\mathrm{e}^{-(3/2)\xi}}{1-\mathrm{e}^{-2\xi}} = \frac{\pi}{4}\frac{\mathrm{e}^{-(\xi/2)}}{\sinh(\xi)},\\
\sum_{n=1}^{\infty}\frac{\mathrm{e}^{-c|\tau|j_n/\ell}}{[J_1(j_n)]^2} &= \sum_{n=1}^{\infty}\frac{\mathrm{e}^{-j_n(2/\pi)\xi}}{[J_1(j_n)]^2}\\
&= -\frac{\pi}{2}\frac{\mathrm{d}}{\mathrm{d}\xi}\sum_{n=1}^{\infty}\frac{\mathrm{e}^{-j_n(2/\pi)\xi}}{j_n[J_1(j_n)]^2}\\
&\approx \frac{\pi^2}{2}\sum_{n=1}^{\infty}(n-1/4)\mathrm{e}^{-(n-1/4)2\xi}\\
&= \frac{\pi}{2}\frac{\pi}{4}\left[\frac{1}{2}\frac{\mathrm{e}^{-(\xi/2)}}{\sinh(\xi)} + \frac{\mathrm{e}^{-\xi/2}}{\sinh^2(\xi)}\cosh(\xi)\right].
\end{aligned}$$

From (49), we can write for small τ

$$\ell f(c\tau,\ell) \approx \frac{\pi}{4}\left\{\frac{2}{\xi} - \frac{\mathrm{e}^{-\xi/2}}{\sinh\xi} - \xi\left[\frac{1}{2}\frac{\mathrm{e}^{-\xi/2}}{\sinh\xi} + \frac{\mathrm{e}^{-\xi/2}}{\sinh^2\xi}\cosh\xi\right]\right\}$$

$$\approx \frac{\pi}{4}\left\{\frac{2}{\xi} - \frac{\mathrm{e}^{-\xi/2}}{\sinh\xi}[1+\xi\coth\xi] - \frac{\xi}{2}\frac{\mathrm{e}^{-\xi/2}}{\sinh\xi}\right\}.$$

The formula (49) can then be expressed in a way which has faster convergence:

$$f(c\tau,\ell) = \frac{1}{c|\tau|} - \frac{1}{\ell}\sum_{n=1}^{\infty}\left\{\frac{\mathrm{e}^{-c|\tau|j_n/\ell}}{j_n[J_1(j_n)]^2} - \frac{\pi}{2}\mathrm{e}^{-(n-1/4)\pi c|\tau|/\ell}\right\}$$
$$- \frac{c|\tau|}{\ell^2}\sum_{n=1}^{\infty}\left\{\frac{\mathrm{e}^{-c|\tau|j_n/\ell}}{[J_1(j_n)]^2} - \frac{\pi^2}{2}(n-1/4)\mathrm{e}^{-(n-1/4)\pi c|\tau|/\ell}\right\}$$
$$- \frac{1}{\ell}\frac{\pi}{4}\frac{\mathrm{e}^{-(\pi/4)c|\tau|/\ell}}{\sinh\left(\frac{\pi}{2}c|\tau|/\ell\right)} - \frac{c|\tau|}{\ell^2}\frac{\pi}{2}\frac{\pi}{4}\left[\frac{1}{2}\frac{\mathrm{e}^{-(\pi/4)c|\tau|/\ell}}{\sinh\left(\frac{\pi}{2}c|\tau|/\ell\right)}\right.$$
$$\left. + \frac{\mathrm{e}^{-(\pi/4)c|\tau|/\ell}}{\sinh^2\left(\frac{\pi}{2}c|\tau|/\ell\right)}\cosh\left(\frac{\pi}{2}c|\tau|/\ell\right)\right]. \quad (52)$$

For numerical computations, the difference of the wake potentials (50) may thus be better rewritten as

$$W^+(\tau) - W^-(\tau) = \frac{1}{2\pi\epsilon_0|\beta|}\frac{\pi}{2}$$
$$\times\left\{\frac{1}{\ell}\left\langle\sum_{n=1}^{\infty}\left[\frac{2}{\pi}\frac{\mathrm{e}^{-c|\tau|j_n/\ell}}{j_n[J_1(j_n)]^2} - \mathrm{e}^{-(n-1/4)\pi c|\tau|/\ell}\right]\right.\right.$$
$$+ \frac{c|\tau|}{\ell}\sum_{n=1}^{\infty}\left[\frac{2}{\pi}\frac{\mathrm{e}^{-c|\tau|j_n/\ell}}{[J_1(j_n)]^2} - \pi(n-1/4)\mathrm{e}^{-(n-1/4)\pi c|\tau|/\ell}\right]$$
$$+ \frac{1}{2}\frac{\mathrm{e}^{-(\pi/4)c|\tau|/\ell}}{\sinh\left(\frac{\pi}{2}c|\tau|/\ell\right)} + \frac{c|\tau|}{\ell}\frac{\pi}{4}\left[\frac{1}{2}\frac{\mathrm{e}^{-(\pi/4)c|\tau|/\ell}}{\sinh\left(\frac{\pi}{2}c|\tau|/\ell\right)}\right.$$
$$\left.\left.\left. + \frac{\mathrm{e}^{-(\pi/4)c|\tau|/\ell}}{\sinh^2\left(\frac{\pi}{2}c|\tau|/\ell\right)}\cosh\left(\frac{\pi}{2}c|\tau|/\ell\right)\right]\right\rangle\right\}_{\ell=a/|\beta\gamma|}^{\ell=b/|\beta\gamma|}. \quad (53)$$

The $1/(c|\tau|)$ term of $f(c\tau,\ell)$ in (52) has disappeared in (53) because it does not depend on ℓ.

Figure 5 shows a comparison between the formula (53) and the results obtained from a mode matching code [10] by a Fourier transformation of the complex impedances[3]. It should also be noted that the wake potentials are normalized to $1/(\epsilon_0 a)$ in [10], whereas in the present paper they are normalized to $1/(4\pi\epsilon_0 a)$. The relative error becomes large when $|s| > 0.5$ because the absolute values become very small.

The value of the difference of the wake potentials for $\tau = 0$ can be evaluated in a simple way by using (38), (40) and (44):

$$W^-(0) - W^+(0) = \frac{1}{2\pi\epsilon_0|\beta|}\left\{\frac{1}{\ell}\right\}_{\ell=a/|\beta\gamma|}^{\ell=b/|\beta\gamma|}\frac{A}{2}$$

[3] It should be noted that in [10], there is an error in the scale of the horizontal axis of Figs. 32 and 33: the published scale should be multiplied by 1.407195

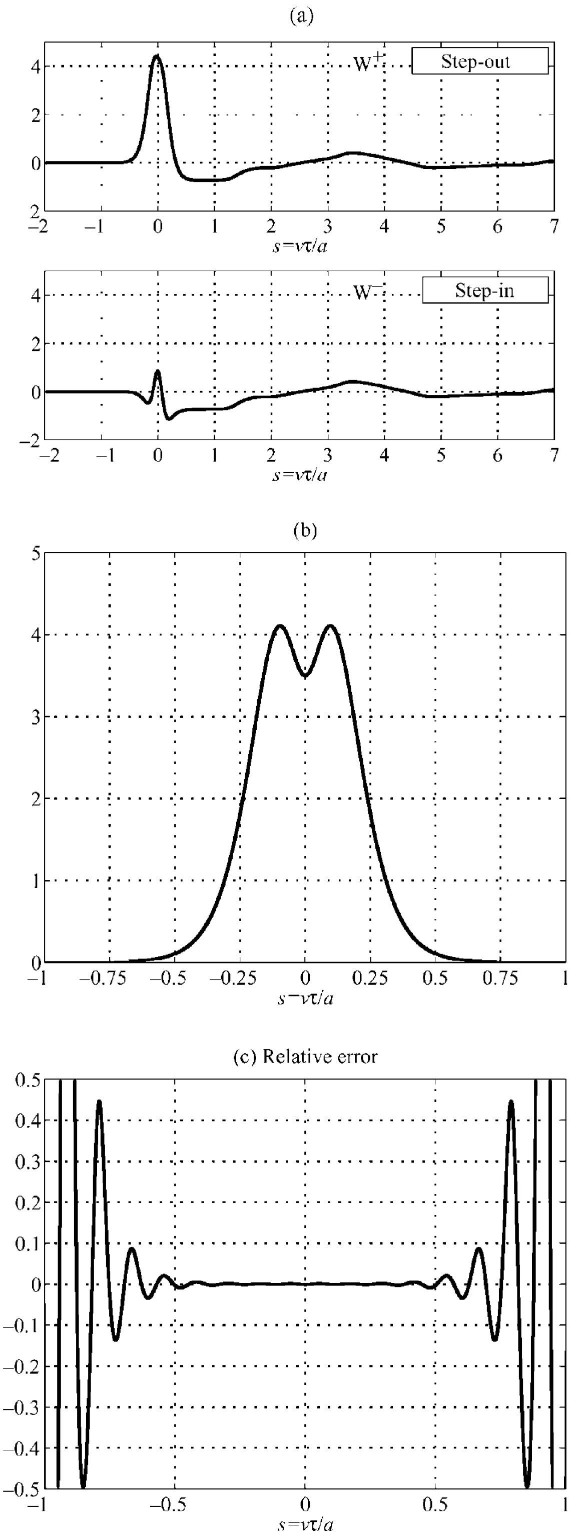

Fig. 5. (**a**) The normalized wake potentials [normalized to $1/(4\pi\varepsilon_0 a)$] for a step between two circular waveguides, evaluated by a mode matching code, with $N_a + N_b = 362$. The Fourier transform of the impedances is computed with 9000 points. (**b**) The difference of the wake potentials $W^+ - W^-$ evaluated by the analytical formula (53) [normalized to $1/(4\pi\varepsilon_0 a)$]. The value at the origin is $\gamma\,(1-(a/b))\,A$. (**c**) The relative error between (**a**) and (**b**), computed as $2((a)-(b))/((a)+(b))$. In all cases $\beta\gamma = 10$ and $a/b = 0.6$

$$= -\frac{\gamma}{4\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)A. \quad (54)$$

6.1 Physical interpretation of expression (47)

The expression (47) can be rewritten as

$$u(c\tau,\ell) = -\frac{\pi}{2}\mathrm{sgn}(c\tau)\zeta\left\{2\sum_{n=1}^{\infty}\frac{\mathrm{e}^{-j_n\zeta}}{j_n[J_0(j_n)]^2}-\frac{1}{\zeta}\right\}, \quad (55)$$

where $\zeta = |c\tau/\ell|$.

This should be compared to the electrostatic potential [15] produced by a point charge q placed at the origin on the axis of a circular conducting cylinder of radius a:

$$V(r,z) = \frac{q}{4\pi\varepsilon_0 a}\left\{2\sum_{n=1}^{\infty}\mathrm{e}^{-j_n|z/a|}\frac{J_0(j_n r/a)}{j_n[J_0(j_n)]^2} - \frac{a}{\sqrt{r^2+z^2}}\right\}. \quad (56)$$

In (56) the first term represents the total potential [15], from which we have subtracted (second term) the potential of the charge q in free space, so that the difference $V(r,z)$ is the potential produced by the image charges on the cylinder wall; the latter potential stays finite at the origin $r = z = 0$. The potential $V(0,z)$ produced on the cylinder axis by the image charges on the wall can be obtained by comparing (56) with $r = 0$ to (55); it is related to the function $u(c\tau,\ell)$ by

$$V(0,z) = \frac{q}{4\pi\varepsilon_0 a}\left[-\frac{2}{\pi}a\frac{u(z,a)}{z}\right].$$

The value of $V(0,0)$ thus reads

$$V(0,0) = \frac{q}{4\pi\varepsilon_0 a}\lim_{\tau\to 0}\left[-\frac{2}{\pi}\ell\frac{u(c\tau,\ell)}{c\tau}\right] = -\frac{q}{4\pi\varepsilon_0 a}\underbrace{\frac{2}{\pi}\int_0^{\infty}\frac{\mathrm{d}x}{[I_0(x)]^2}}_{A=0.870690} \quad (57)$$

from (44) and (45). In (57), $-(q/4\pi\varepsilon_0 a)$ would be the potential produced at the origin by an image charge $-q$ concentrated in the plane $z = 0$ on the cylinder. Since in reality the image charge is not concentrated at $z = 0$, this potential is reduced by a factor $A < 1$.

Note that the electrostatic energy of the charge q at rest in the cylinder of radius a is, apart from its self-energy,

$$\frac{1}{2}qV(0,0) = -\frac{q^2}{4\pi\varepsilon_0 a}\frac{A}{2}. \quad (58)$$

If the charge is moving with velocity v on the axis of the cylinder, its total electromagnetic energy equals the integral from $t = -\infty$ to $+\infty$ of the flux of the Poynting vector in time domain through a cross-section of the waveguide. By Parseval's theorem, this integral is equal to the inverse Fourier transform at $\tau = 0$ of the flux of the complex Poynting vector in frequency domain through a cross-section of the waveguide:

$$\int_{S_a}[\mathbf{E}^{0+}\times(\mathbf{H}^{0+})^*]\cdot\hat{z}\mathrm{d}S = \int_{S_a}\mathrm{Re}[\mathbf{E}^{0+}\times(\mathbf{H}^{0+})^*]\cdot\hat{z}\mathrm{d}S. \quad (59)$$

We may restrict ourselves to the real part of this expression because the total integral must be real. Again, as in (32), we must subtract from the integral the infinite power flux of the point charge in free space, which corresponds to the self-energy of the charge, and which disappears in the difference (19). Since the integral (59) appears, multiplied by a factor 2, in (19), we can obtain its inverse Fourier transform at $\tau = 0$ by taking half of the result which appears in (54) for waveguide a:

$$-q^2\frac{\gamma}{4\pi\varepsilon_0 a}\frac{A}{2} = \gamma\frac{1}{2}qV(0,0). \quad (60)$$

Again, the self-energy of the charge q is not counted in this expression. In (60), we verify the relativistic transformation of the rest energy (58) through the factor γ [16].

6.2 Physical interpretation of expression (54)

Let us rewrite expression (54) as

$$W^+(0) - W^-(0) = \frac{\gamma}{4\pi\varepsilon_0}A\left(\frac{1}{a}-\frac{1}{b}\right). \quad (61)$$

Since $qW(\tau) = -\mathrm{sgn}(v)\int_{-\infty}^{+\infty}E_z^s(r=0,z,z/v+\tau)\mathrm{d}z$, it follows that $q^2W(0)$ represents the total energy lost by the particle from $t = -\infty$ to $t = \infty$. This energy is made up of two parts: one part is used to compensate for the energy change in the image charges of the charge q when going from the input waveguide to the output waveguide; the second part is radiated from the scatterer. Using (58) and (60) we obtain

$$W^+(0) = \frac{\gamma}{4\pi\varepsilon_0}\frac{A}{2}\left(-\frac{1}{b}+\frac{1}{a}\right)+\frac{1}{q^2}\mathcal{E}^+_{\mathrm{rad}},$$
$$W^-(0) = \frac{\gamma}{4\pi\varepsilon_0}\frac{A}{2}\left(-\frac{1}{a}+\frac{1}{b}\right)+\frac{1}{q^2}\mathcal{E}^-_{\mathrm{rad}}, \quad (62)$$

which yields

$$W^+(0) - W^-(0) = \frac{\gamma}{4\pi\varepsilon_0}A\left(\frac{1}{a}-\frac{1}{b}\right)+\frac{1}{q^2}(\mathcal{E}^+_{\mathrm{rad}}-\mathcal{E}^-_{\mathrm{rad}}). \quad (63)$$

The comparison with (61) shows that $\mathcal{E}^-_{\mathrm{rad}} = \mathcal{E}^+_{\mathrm{rad}}$: the amount of energy radiated from the scatterer does not depend on the direction of motion of the particle.

When $|\beta\gamma| = \infty$, causality requires that $W^-(\tau) = W^+(\tau) = 0$ for $\tau < 0$. Since $[W^-(\tau) - W^+(\tau)]$ is an even function of τ, this difference must vanish everywhere except at $\tau = 0$. We verify that when $|\beta\gamma| = \infty$, (38) reduces to

$$W^+(\tau) - W^-(\tau) = \frac{1}{2\pi\epsilon_0|\beta|}2\ln\left(\frac{b}{a}\right)\delta(c\tau)$$

$$= \frac{1}{\pi\epsilon_0 a} \ln\left(\frac{b}{a}\right) \delta\left(\frac{v\tau}{a}\right). \tag{64}$$

Such a behavior is already apparent in Fig. 5(b,) which shows the case $a/b = 0.6$, $\beta\gamma = 10$.

7 Conclusions

After having derived once again the difference of the coupling impedances for particles with opposite velocities by using the reciprocity theorem, we apply the resulting formulae to the case of two circular waveguides of different radii, for which we compute explicitly the difference of the coupling impedances and of the wake potentials.

Appendix

A An explicit proof of formula (17)

In this appendix we give a complete proof that the difference of the coupling impedances is independent from the scattered fields in the waveguides, namely of the conjecture presented in Sect. 4. Let us start from the left-hand side of (2):

$$\begin{aligned}
&[\mathbf{E}^{+} \times \mathbf{H}^{-}]_z - [\mathbf{E}^{-} \times \mathbf{H}^{+}]_z \\
&= \overbrace{[\mathbf{E}^{0+} \times \mathbf{H}^{0-}]_z - [\mathbf{E}^{0-} \times \mathbf{H}^{0+}]_z}^{\mathbf{S}^0} \\
&+ \overbrace{[\mathbf{E}^{\mathrm{s}+} \times \mathbf{H}^{\mathrm{s}-}]_z - [\mathbf{E}^{\mathrm{s}-} \times \mathbf{H}^{\mathrm{s}+}]_z}^{\mathbf{S}^{\mathrm{s}}} \\
&+ \underbrace{[\mathbf{E}^{0+} \times \mathbf{H}^{\mathrm{s}-}]_z - [\mathbf{E}^{\mathrm{s}-} \times \mathbf{H}^{0+}]_z + [\mathbf{E}^{\mathrm{s}+} \times \mathbf{H}^{0-}]_z - [\mathbf{E}^{0-} \times \mathbf{H}^{\mathrm{s}+}]_z}_{\mathbf{S}^{0\mathrm{s}}}.
\end{aligned}$$

The $\mathbf{S}^0$ term does not depend on z. When integrated over S, the $\mathbf{S}^{\mathrm{s}}$ term disappears because of the orthogonality relations (16) in (14).

About the $\mathbf{S}^{0\mathrm{s}}$ term, let us remember that the primary field is an E mode. Therefore we have [17]

$$\begin{aligned}
\mathbf{E}_{\perp}^{0+} &= \frac{jk}{\beta\kappa^2}\mathrm{grad}_{\perp} E_z^{0+}, \\
\zeta_0 \mathbf{H}_{\perp}^{0+} &= -\beta[\mathbf{E}_{\perp}^{0+} \times \hat{z}], \\
\mathbf{E}_{\perp}^{0-} &= -\frac{jk}{\beta\kappa^2}\mathrm{grad}_{\perp} E_z^{0-}, \\
\zeta_0 \mathbf{H}_{\perp}^{0-} &= \beta[\mathbf{E}_{\perp}^{0-} \times \hat{z}].
\end{aligned} \tag{A.1}$$

In waveguide a we may rewrite the relations (4) to (7) as

$$\begin{aligned}
\mathbf{E}_{\perp}^{\mathrm{s}\pm} &= \sum_n a_n^{\pm} \mathbf{E}_n \mathrm{e}^{-jk_n|z|}, \\
\mathbf{H}_{\perp}^{\mathrm{s}\pm} &= \sum_n a_n^{\pm} \mathbf{H}_n \mathrm{e}^{-jk_n|z|}, \\
E_z^{\mathrm{s}\pm} &= \sum_n a_n^{\pm} E_{nz} \mathrm{e}^{-jk_n|z|},
\end{aligned}$$

$$H_z^{\mathrm{s}\pm} = \sum_n a_n^{\pm} H_{nz} \mathrm{e}^{-jk_n|z|}. \tag{A.2}$$

The same relations apply to waveguide b, with $a_n^{\pm}$ replaced by $b_n^{\pm}$.

Also, the following relations [17] apply between the transverse modal vectors of a generic waveguide:

$$\text{for } E \text{ modes:} \quad H_{nz} = 0, \quad \zeta_0 \mathbf{H}_n = -\mathrm{sgn}(z)\frac{k}{k_n}[\mathbf{E}_n \times \hat{z}],$$
$$\mathbf{E}_n = -\mathrm{sgn}(z)\frac{jk_n}{k_{cn}^2}\mathrm{grad}_{\perp} E_{nz}, \tag{A.3}$$

$$\text{for } H \text{ modes:} \quad E_{nz} = 0, \quad \mathbf{E}_n = \mathrm{sgn}(z)\zeta_0\frac{k}{k_n}[\mathbf{H}_n \times \hat{z}],$$
$$\mathbf{H}_n = -\mathrm{sgn}(z)\frac{jk_n}{k_{cn}^2}\mathrm{grad}_{\perp} H_{nz}, \tag{A.4}$$

where $k_{cn}^2 = k^2 - k_n^2$.

Let us now consider the E modes and the term $[\mathbf{E}^{0+} \times \mathbf{H}^{\mathrm{s}-}]_z$ of $\mathbf{S}^{0\mathrm{s}}$:

$$\begin{aligned}
[\mathbf{E}^{0+} \times \mathbf{H}^{\mathrm{s}-}]_z &= \sum_n a_n^{-} \mathrm{e}^{-jk_n|z|}[\mathbf{E}_{\perp}^{0+} \times \mathbf{H}_n]_z \\
&= -\frac{1}{\zeta_0}\mathrm{sgn}(z)\sum_n a_n^{-} \mathrm{e}^{-jk_n|z|}\frac{k}{k_n}[\mathbf{E}_{\perp}^{0+} \times [\mathbf{E}_n \times \hat{z}]]_z \\
&= \frac{1}{\zeta_0}\mathrm{sgn}(z)\sum_n a_n^{-} \mathrm{e}^{-jk_n|z|}\frac{k}{k_n}(\mathbf{E}_{\perp}^{0+} \cdot \mathbf{E}_n).
\end{aligned} \tag{A.5}$$

With (A.3) and (A.1) the term $(\mathbf{E}_n \cdot \mathbf{E}_{\perp}^{0+})$ can be written as

$$(\mathbf{E}_n \cdot \mathbf{E}_{\perp}^{0+}) = -\mathrm{sgn}(z)\frac{jk_n}{k_{cn}^2}\frac{jk}{\beta\kappa^2}(\mathrm{grad}_{\perp} E_{nz} \cdot \mathrm{grad}_{\perp} E_z^{0+}), \tag{A.6}$$

where the fields E_z^{0+} and E_{nz} verify the Helmholtz equations:

$$\triangle_{\perp} E_z^{0\pm} - \kappa^2 E_z^{0\pm} = 0, \quad \triangle_{\perp} E_{nz} + k_{cn}^2 E_{nz} = 0,$$

whence

$$\begin{aligned}
&\mathrm{div}(E_z^{0+}\mathrm{grad}_{\perp} E_{nz}) \\
&\quad = \mathrm{grad}_{\perp} E_z^{0+} \cdot \mathrm{grad}_{\perp} E_{nz} - k_{cn}^2 E_{nz} E_z^{0+}, \\
&\mathrm{div}(E_{nz}\mathrm{grad}_{\perp} E_z^{0+}) \\
&\quad = \mathrm{grad}_{\perp} E_{nz} \cdot \mathrm{grad}_{\perp} E_z^{0+} + \kappa^2 E_z^{0+} E_{nz}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\kappa^2 \mathrm{div}(E_z^{0+}\mathrm{grad}_{\perp} E_{nz}) + k_{cn}^2 \mathrm{div}(E_{nz}\mathrm{grad}_{\perp} E_z^{0+}) \\
&\quad = (\kappa^2 + k_{cn}^2)(\mathrm{grad}_{\perp} E_{nz} \cdot \mathrm{grad}_{\perp} E_z^{0+}),
\end{aligned}$$

which can be integrated over the transverse cross-section:

$$\begin{aligned}
&(\kappa^2 + k_{cn}^2)\int_S (\mathrm{grad}_{\perp} E_{nz} \cdot \mathrm{grad}_{\perp} E_z^{0+})\mathrm{d}S \\
&= \pm\kappa^2 \oint_{\partial S} E_z^{0+}\frac{\partial E_{nz}}{\partial r}\mathrm{d}s \pm k_{cn}^2 \oint_{\partial S} E_{nz}\frac{\partial E_z^{0+}}{\partial r}\mathrm{d}s,
\end{aligned} \tag{A.7}$$

where the line integral over ∂S must also include a small circle around the axis ($r=0$) where the primary field is singular and $\mathrm{d}s = r\mathrm{d}\varphi$. The $\pm$ sign should be taken as + on the waveguide wall, and as − on the small circle around the z-axis.

The first integral in the right-hand side of (A.7) is zero, since on the contour of the transverse cross-section E_z^{0+} is equal to zero, and since on the circle around the axis the primary field behaves as $\log(\kappa r)$ (it is the free-space term $K_0(\kappa r)$); the limit for $r \to 0$ of the product $r\log(r)$ is zero. The second integral is equal to zero on the waveguide wall where $E_{nz}=0$; the contribution of the small circle around the z-axis is, with (20),

$$-k_{cn}^2 E_{nz}(r=0) q\zeta_0 \frac{\kappa^2}{jk} \mathrm{e}^{-j(k/\beta)z}, \quad \beta > 0,$$

since the primary field E_z^{0+} contains a free-space term $K_0(\kappa r)$ whose radial derivative behaves as $1/r$ when $r \to 0$, and the limit of its product by r is finite. Therefore, (A.7) yields

$$\int_S (\mathrm{grad}_\perp E_{nz} \cdot \mathrm{grad}_\perp E_z^{0+}) \mathrm{d}S = -q\zeta_0 \frac{k_{cn}^2}{\kappa^2 + k_{cn}^2} \frac{\kappa^2}{jk} E_{nz}(r=0) \mathrm{e}^{-j(k/\beta)z},$$

where $\beta > 0$. Finally, gathering (A.5), (A.6), and (A.7), the integral over S of the term $[\mathbf{E}^{0+} \times \mathbf{H}^{\mathrm{s}-}]_z$ of $\mathbf{S}^{0\mathrm{s}}$ reads

$$\int_S [\mathbf{E}^{0+} \times \mathbf{H}^{\mathrm{s}-}]_z \mathrm{d}S = \frac{q}{\beta} \sum_n \frac{jk}{\kappa^2 + k_{cn}^2} a_n^- E_{nz}(r=0) \mathrm{e}^{-jk_n|z| - j(k/\beta)z}. \tag{A.8}$$

Let us now consider the term $[\mathbf{E}^{\mathrm{s}-} \times \mathbf{H}^{0+}]_z$ of $\mathbf{S}^{0\mathrm{s}}$. With (A.2) and (A.1) we have

$$[\mathbf{E}^{\mathrm{s}-} \times \mathbf{H}^{0+}]_z = -\frac{\beta}{\zeta_0} \sum_n a_n^- \mathrm{e}^{-jk_n|z|} [\mathbf{E}_n \times [\mathbf{E}_\perp^{0+} \times \hat{z}]]_z = \frac{\beta}{\zeta_0} \sum_n a_n^- \mathrm{e}^{-jk_n|z|} (\mathbf{E}_n \cdot \mathbf{E}_\perp^{0+}).$$

By analogy with (A.8) the integral of this term over S is

$$\int_S [\mathbf{E}^{\mathrm{s}-} \times \mathbf{H}^{0+}]_z \mathrm{d}S = q\,\mathrm{sgn}(z) \sum_n \frac{jk_n}{\kappa^2 + k_{cn}^2} a_n^- E_{nz}(r=0) \mathrm{e}^{-jk_n|z| - j(k/\beta)z}. \tag{A.9}$$

With (A.2) and (A.1) the term $[\mathbf{E}^{\mathrm{s}+} \times \mathbf{H}^{0-}]_z$ becomes

$$[\mathbf{E}^{\mathrm{s}+} \times \mathbf{H}^{0-}]_z = \frac{\beta}{\zeta_0} \sum_n a_n^+ \mathrm{e}^{-jk_n|z|} [\mathbf{E}_n \times [\mathbf{E}_\perp^{0-} \times \hat{z}]]_z = -\frac{\beta}{\zeta_0} \sum_n a_n^+ \mathrm{e}^{-jk_n|z|} (\mathbf{E}_n \cdot \mathbf{E}_\perp^{0-}). \tag{A.10}$$

Comparing with (A.5), we see that $\mathbf{E}_\perp^{0+}$ has been replaced by $\mathbf{E}_\perp^{0-}$; from (8) and (10) this only means replacing $\mathrm{e}^{-j(k/\beta)z}$ by $\mathrm{e}^{j(k/\beta)z}$. Therefore, by analogy with (A.8), the integral over S of (A.10) reads

$$\int_S [\mathbf{E}^{\mathrm{s}+} \times \mathbf{H}^{0-}]_z \mathrm{d}S = -q\,\mathrm{sgn}(z) \sum_n \frac{jk_n}{\kappa^2 + k_{cn}^2} a_n^+ E_{nz}(r=0) \mathrm{e}^{-jk_n|z| + j(k/\beta)z}, \quad \beta > 0. \tag{A.11}$$

For the last term $[\mathbf{E}^{0-} \times \mathbf{H}^{\mathrm{s}+}]_z$ of the E modes,

$$[\mathbf{E}^{0-} \times \mathbf{H}^{\mathrm{s}+}]_z = -\frac{\mathrm{sgn}(z)}{\zeta_0} \sum_n a_n^+ \mathrm{e}^{-jk_n|z|} \frac{k}{k_n} [\mathbf{E}_\perp^{0-} \times (\mathbf{E}_n \times \hat{z})]_z = \frac{\mathrm{sgn}(z)}{\zeta_0} \sum_n a_n^+ \mathrm{e}^{-jk_n|z|} \frac{k}{k_n} (\mathbf{E}_n \cdot \mathbf{E}_\perp^{0-}),$$

and the integral of this term over S is

$$\int_S [\mathbf{E}^{0-} \times \mathbf{H}^{\mathrm{s}+}]_z \mathrm{d}S = \frac{q}{\beta} \sum_n \frac{jk}{\kappa^2 + k_{cn}^2} a_n^+ E_{nz}(r=0) \mathrm{e}^{-jk_n|z| + j(k/\beta)z}. \tag{A.12}$$

Finally, the whole integral of the $\mathbf{S}^{0\mathrm{s}}$ term reads, for the E modes,

$$\int_S \mathbf{S}^{0\mathrm{s}} \mathrm{d}S = q \sum_n \frac{j}{\kappa^2 + k_{cn}^2} \left[\frac{k}{\beta} - \mathrm{sgn}(z) k_n\right] \times a_n^- E_{nz}(r=0) \mathrm{e}^{-jk_n|z| - j(k/\beta)z} + q \sum_n \frac{j}{\kappa^2 + k_{cn}^2} \left[-\frac{k}{\beta} - \mathrm{sgn}(z) k_n\right] \times a_n^+ E_{nz}(r=0) \mathrm{e}^{-jk_n|z| + j(k/\beta)z}. \tag{A.13}$$

In the left-hand side of (2), because of the direction of the outer normal $\hat{n}$, this integral over the waveguide cross-section must be taken with a −sign at z_1 and with a +sign at z_2.

We will now show that this expression taken at z_1 or z_2 is identical to the corresponding exponential terms in (12), which is a form of the right-hand side of (2), and therefore they exactly cancel each other. For example, the indefinite integral of $E_z^{\mathrm{s}-}$ in (12) reads, with (A.2),

$$q \int E_z^{\mathrm{s}-}(r=0,z) \mathrm{e}^{-j(k/\beta)z} \mathrm{d}z = q \sum_n a_n^- E_{nz}(r=0) \int \mathrm{e}^{-jk_n|z|} \mathrm{e}^{-j(k/\beta)z} \mathrm{d}z = q \sum_n a_n^- E_{nz}(r=0) \frac{j}{\frac{k}{\beta} + \mathrm{sgn}(z) k_n} \mathrm{e}^{-jk_n|z| - j(k/\beta)z} = q \sum_n a_n^- E_{nz}(r=0) \left[\frac{k}{\beta} - \mathrm{sgn}(z) k_n\right]$$

$$\times \frac{j}{\kappa^2 + k_{cn}^2} \mathrm{e}^{-jk_n|z| - j(k/\beta)z}$$

because $k_{cn}^2 = k^2 - k_n^2$. This expression is identical to the first term of (A.13); in (12), it must also be taken with a −sign at z_1 and with a +sign at z_2. The same applies to the integral of E_z^{s+} in (12). Therefore, the definite integrals cancel each other exactly in (2).

Let us now consider the H modes and the terms $[\mathbf{E}_\perp^{0+} \times \mathbf{H}_n]_z$ which appear in $[\mathbf{E}^{0+} \times \mathbf{H}^{s-}]_z$ of $\mathbf{S}^{0s}$. With (A.1),

$$\begin{aligned}[\mathbf{E}_\perp^{0+} \times \mathbf{H}_n]_z &= \frac{jk}{\beta\kappa^2}[\mathrm{grad}_\perp E_z^{0+} \times \mathbf{H}_n]_z \\ &= \frac{jk}{\beta\kappa^2}\left\{\mathrm{curl}(E_z^{0+}\mathbf{H}_n) - E_z^{0+}\mathrm{curl}(\mathbf{H}_n)\right\}_z .\end{aligned}$$

From (A.4), the last term is equal to zero for the H modes. As a consequence, with only the first term left we have

$$\begin{aligned}\int_S [\mathrm{grad}_\perp E_z^{0+} \times \mathbf{H}_n]_z \mathrm{d}S &= \int_S \mathrm{curl}_z(E_z^{0+}\mathbf{H}_n)\mathrm{d}S \\ &= \oint_{\partial S} E_z^{0+}\mathbf{H}_n \cdot \mathrm{d}\mathbf{s},\end{aligned}$$

which is zero for the same reasons as the first term in the right-hand side of (A.7). Therefore

$$\int_S [\mathbf{E}_\perp^{0+} \times \mathbf{H}_n]_z \mathrm{d}S = 0. \tag{A.14}$$

Similarly

$$\int_S [\mathbf{E}_\perp^{0-} \times \mathbf{H}_n]_z \mathrm{d}S = 0.$$

Consider now the terms $[\mathbf{E}_n \times \mathbf{H}^{0+}]_z$ which appear in $[\mathbf{E}^{s-} \times \mathbf{H}^{0+}]_z$ of $\mathbf{S}^{0s}$; from (A.1) and (A.4) we obtain

$$\begin{aligned}[\mathbf{E}_n \times \mathbf{H}^{0+}]_z &= -\frac{\beta}{\zeta_0}[\mathbf{E}_n \times [\mathbf{E}_\perp^{0+} \times \hat{z}]]_z = \frac{\beta}{\zeta_0}(\mathbf{E}_n \cdot \mathbf{E}_\perp^{0+}) \\ &= \mathrm{sgn}(z)\beta\frac{k}{k_n}\mathbf{E}_\perp^{0+} \cdot [\mathbf{H}_n \times \hat{z}] \\ &= \mathrm{sgn}(z)\beta\frac{k}{k_n}[\mathbf{E}_\perp^{0+} \times \mathbf{H}_n]_z.\end{aligned}$$

Therefore, with (A.14),

$$\int_S [\mathbf{E}_n \times \mathbf{H}^{0+}]_z \mathrm{d}S = 0. \tag{A.15}$$

Similarly

$$\int_S [\mathbf{E}_n \times \mathbf{H}^{0-}]_z \mathrm{d}S = 0.$$

Finally, due to (A.14) and (A.15), the contribution of $\mathbf{S}^{0s}$ vanishes for the H modes, as it should because the H modes do not contribute to (12).

B Another derivation of formula (33)

Let us now derive the expression (33) by using another method. From (17) and (30) we can write

$$Z_\parallel^-(k) - Z_\parallel^+(k) = \frac{\zeta_0}{2\pi|\beta|}[\Im(b) - \Im(a)], \tag{B.1}$$

where

$$\Im(\ell) = \lim_{r\to 0}\int_{\kappa r}^{\kappa\ell} - \left[K_1(x) + I_1(x)\frac{K_0(\kappa\ell)}{I_0(\kappa\ell)}\right]^2 2x\mathrm{d}x. \tag{B.2}$$

Let us introduce the auxiliary function

$$G(x) = K_0(x) - I_0(x)\frac{K_0(\kappa\ell)}{I_0(\kappa\ell)}, \quad \text{with} \quad G(\kappa\ell) = 0. \tag{B.3}$$

It is easy to verify the following properties:

$$\begin{aligned}G'(x) &= -\left[K_1(x) + I_1(x)\frac{K_0(\kappa\ell)}{I_0(\kappa\ell)}\right], \\ G'(\kappa\ell) &= -\frac{1}{\kappa\ell I_0(\kappa\ell)}, \\ \int (G')^2 2x\mathrm{d}x &= 2xG'G - 2\int G\frac{\mathrm{d}}{\mathrm{d}x}(xG')\mathrm{d}x \\ &= 2xGG' - 2\int G^2 x\mathrm{d}x \\ &= 2xGG' - x^2G^2 + 2\int (xG)(xG')\mathrm{d}x \\ &= 2xGG' - x^2G^2 + (xG')^2\end{aligned}$$

because the Bessel differential equation for functions of order zero yields $(\mathrm{d}/\mathrm{d}x)(xG') = xG$. Therefore, the equation (B.2) can also be written as

$$\begin{aligned}\Im(\ell) = \lim_{r\to 0}&\Bigg\{-\frac{1}{[I_0(\kappa\ell)]^2} + 2\kappa r G(\kappa r)G'(\kappa r) \\ &- [\kappa r G(\kappa r)]^2 + [\kappa r G'(\kappa r)]^2\Bigg\} \\ =& -\frac{1}{[I_0(\kappa\ell)]^2} \\ &- \lim_{r\to 0} 2\left[K_0(\kappa r) - I_0(\kappa r)\frac{K_0(\kappa\ell)}{I_0(\kappa\ell)}\right] + 1\end{aligned}$$

because $\lim_{r\to 0}\kappa r G'(\kappa r) = -1$,

$$= -\frac{1}{[I_0(\kappa\ell)]^2} + 1 - \lim_{r\to 0} 2K_0(\kappa r) + 2\frac{K_0(\kappa\ell)}{I_0(\kappa\ell)}.$$

The radius r of the infinitely thin cylinder around the beam must be the same in both waveguides; therefore the infinite $K_0(\kappa r)$ term which does not depend on ℓ, disappears in the difference (B.1).

Finally

$$\Im(b) - \Im(a) = \left\{-\frac{1}{[I_0(\kappa\ell)]^2} + 2\frac{K_0(\kappa\ell)}{I_0(\kappa\ell)}\right\}_{\ell=a}^{\ell=b}$$

and

$$Z_{\parallel}^{-}(k) - Z_{\parallel}^{+}(k) = \frac{\zeta_0}{2\pi|\beta|} \left\{ \frac{2K_0(\kappa\ell)}{I_0(\kappa\ell)} - \frac{1}{[I_0(\kappa\ell)]^2} \right\}_{\ell=a}^{\ell=b}, \tag{B.4}$$

which is identical to (33).

C Proof of the expansion (46)

Let us now give some details for a possible derivation of (46). In [18], Whittaker and Watson derive an expansion in rational functions, valid for a function whose singularities are only simple poles, provided the function stays finite at points situated between neighboring poles when these points are going to infinity. With C being the Euler constant, we observe that the function $(K_0(x)/I_0(x)) + (\mathrm{C} + \ln(x/2))$ is a function of x^2 which vanishes at $x^2 = 0$ [11] and which has only simple poles as singularities. Between poles, this function goes to infinity like $\ln(x)$ instead of staying finite, but such an infinity is weak enough for the proof of Whittaker and Watson to still apply. Therefore we can write [18]

$$\frac{K_0(x)}{I_0(x)} = -\left(\mathrm{C} + \ln\frac{x}{2}\right) + \sum_{n=1}^{\infty} B_n \left[\frac{1}{j_n^2} - \frac{1}{x^2 + j_n^2}\right], \tag{C.1}$$

where the j_n are the zeros of $J_0(z)$ and the B_n are constant coefficients.

By a Taylor expansion it is easy to see that in the vicinity of a zero $\pm ij_n$ the function $I_0(x)$ behaves as

$$I_0(x) = \frac{J_1(j_n)}{2j_n}(x^2 + j_n^2) + O(x^2 + j_n^2)^2, \tag{C.2}$$

whereas from the Wronskian

$$I_1(x)K_0(x) + I_0(x)K_1(x) = 1/x$$

we obtain, for $x = \pm ij_n$,

$$K_0(x) = \frac{1}{xI_1(x)} = -\frac{1}{j_n J_1(j_n)}. \tag{C.3}$$

Dividing (C.3) by (C.2) we see that in (C.1),

$$B_n = \frac{2}{[J_1(j_n)]^2}. \tag{C.4}$$

Taking the derivative of (C.1) with respect to x yields, after multiplication by $(-x)$:

$$\frac{1}{[I_0(x)]^2} = 1 - 2x^2 \sum_{n=1}^{\infty} \frac{2}{[J_1(j_n)]^2(x^2 + j_n^2)^2}, \tag{C.5}$$

which is the expansion (46).

References

1. L. Palumbo, V.G. Vaccaro, M. Zobov, Wake Fields and Impedance, report LNF–94/041 (P), 5 September 1994, or report CERN 95–06 (1995), pp. 331–390
2. S.A. Heifets, S.A. Kheifets, Rev. Mod. Phys. **63**, 631 (1991)
3. S.A. Kheifets, B.W. Zotter, Impedances and wakes in high-energy particle accelerators (World Scientific, Singapore 1998), p. 93
4. S.A. Heifets, On Directional Symmetry of Impedance, Technical Report SLAC/AP–79, 1990
5. R. Gluckstern, B.W. Zotter, The Coupling Impedance of Asymmetric Cavities, LEP–Note 613, CERN, 1988
6. J.J. Bisognano, On the Independence of the Longitudinal Wakefield on Direction of Transit through a Cavity, report CEBAF-TN-0109, 1990
7. S. Heifets, B. Zotter, On the Symmetry of the Impedance, in Proceedings of the 1999 IEEE Particle Accelerator Conference, New York, 1999, pp. 1118–1119
8. R.E. Collin, Foundations for microwave engineering, second edition (McGraw-Hill International Editions, Singapore 1992), p. 63
9. R.E. Collin, Field theory of guided waves, second edition (IEEE Press, New York 1991), p. 336
10. D. Davino, G. Dôme, G. Miano, G. Panariello, V.G. Vaccaro, L. Verolino, Nuovo Cimento A **112**, 1401 (1999)
11. M. Abramowitz, I. Stegun, Handbook of mathematical functions (Dover Publications Inc., New York 1970)
12. G.N. Watson, A treatise on the theory of Bessel functions, second edition (Cambridge University Press 1966), p. 135
13. R. Mittra, W.W. Lee, Analytical techniques in the theory of guided waves (Macmillan, New York 1971)
14. A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, Higher transcendental functions, vol. 2 (McGraw-Hill, New York 1953), Eq. (63), p. 104
15. W.R. Smythe, Static and dynamic electricity (McGraw-Hill, New York 1950), p. 178
16. J.D. Jackson, Classical electrodynamics, third edition (John Wiley & Sons, New York 1999), p. 759
17. G. Dôme, RF systems: Waveguides and cavities, in AIP Conference Proceedings no. 153, American Institute of Physics, New York, 1987, p. 1310; or RF Theory in CERN Accelerator School: RF Engineering for Particle Accelerators, report CERN 92–03, 1992, p. 36
18. E.T. Whittaker, G.N. Watson, Modern analysis (Cambridge University Press 1952), p. 134